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Permeable conformal walls and holography

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Abstract: We study conformal field theories in two dimensions separated by domain walls, which preserve at least one Virasoro algebra. We develop tools to study such domain walls, extending and clarifying the concept of 'folding' discussed in the condensed-matter literature. We analyze the conditions for unbroken supersymmetry, and discuss the holographic duals in AdS_3 when they exist. One of the interesting observables is the Casimir energy between a wall and an anti-wall. When these separate free scalar field theories with different target-space radii, the Casimir energy is given by the dilogarithm function of the reflection probability. The walls with holographic duals in AdS_3 separate two sigma models, whose target spaces are moduli spaces of Yang-Mills instantons on T^4 or K3. In the supergravity limit, the Casimir energy is computable as classical energy of a brane that connects the walls through AdS_3. We compare this result with expectations from the sigma-model point of view.

Keywords: Boundary Quantum Field Theory, AdS-CFT and dS-CFT Correspondence, D-branes.

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Contents

1. Introduction

Starting with the pioneering work of Cardy [1], boundary conformal field theory (BCFT) has evolved into a rich subject of great physical interest. The subject is of obvious relevance to the study of critical phenomena in statistical mechanics. Furthermore, two-dimensional conformal boundary states have acquired new importance in recent years, as building blocks for the D(irichlet) branes of string theory [2]. The interplay between the algebraic approach of Conformal Field Theory, and the complementary geometric viewpoint of D-branes, has been the theme of many recent investigations (see e.g. [3, 4] and references therein).

The usual setting of BCFT is a space(time) ending on a boundary. In this setting all incident waves are reflected back,[1] because there is nothing they can transmit to on the other side. One may, however, also consider a situation in which two (or more) non-trivial CFT's

[1] The language is somewhat loose, because strictly-speaking a CFT has no asymptotic particle states. A more accurate phrasing, in two dimensions, is that the boundary state maps holomorphic into antiholomorphic fields, in a way that commutes with the action of the Virasoro algebra.

are glued together along a common interface. The interface can be permeable, meaning that incident waves are partly reflected and partly transmitted. Examples of such boundaries (mostly between identical CFT's) have been discussed in the condensed-matter literature, see for instance [5, 6, 7, 8]. One of our purposes in this work will be to analyze such permeable interfaces in general, and from a rather different, more geometric perspective.

Our interest in these questions was motivated by an issue in holography. String theory in AdS_3 has static solutions describing infinitely-long (p, q) strings, which stretch between two points on the AdS_3 boundary [9]. In the dual spacetime CFT [10]–[15] the endpoint of a (p, q) string is, as we will explain, an interface separating regions with different values of the central charge or different values of the moduli. Similar configurations have been also discussed in higher dimensions [16, 17]. The force exerted by the stretched string on its endpoints translates, in the dual interpretation, to the Casimir force between two (or more, if one considers string networks) permeable interfaces. In this paper we will calculate this Casimir force, both in the weak- and in the strong-coupling limits. The results we find are in some ways reminiscent of the heavy quark-antiquark potential in four-dimensional $N = 4$ super Yang-Mills [18, 19].

From a technical point of view, an interface between two CFTs is described by a regular boundary state in the tensor-product theory.[2] This is intuitively obvious since one can 'fold' space along the interface, so that both CFTs live on the same side [5]. Permeable walls, in particular, are simply boundary states of the tensor product, that cannot be expressed in terms of Ishibashi states of the factor theories. Their study does not, therefore, require drastically-new technology, but it leads to a host of novel questions and observables which are not usually considered in the standard BCFT setting. One example of such a new observable is the Casimir energy of a 'CFT bubble' which we calculate.

The plan of this paper is as follows. In section 2 we introduce the main ideas of 'conformal gluing' in the simplest context of a free scalar field theory, and explain how this is related to conventional conformal boundary states. We calculate the Casimir energy for two identical interfaces, separating regions with different target-circle radii, and show that it is given by the dilogarithm function of the reflection probability. In section 3 we generalize these considerations in several directions. We show how superconformal invariance of the walls can be guaranteed by the continuity of appropriately-defined 'half' superfields, in a manifestly supersymmetric formalism. We also calculate the fermionic contribution to the Casimir energy, and then go on to discuss general properties of permeable interfaces and some more examples. In section 4 we turn our attention to interfaces of two-dimensional CFTs which admit holographic AdS_3 duals. We calculate the classical energy of a (p, q) string as a function of its tension, Neveu-Schwarz-charge and of the separation of its two endpoints. We discuss the validity of this calculation, and interpret it as Casimir energy in the dual spacetime sigma model. We point out an intriguing analogy with operator algebras on instanton moduli spaces defined in the mathematics literature [20, 21, 22]. We conclude, in section 5, with some comments on future directions.

[2] More precisely, the tensor product of the theory on one side and of the 'conjugate' theory, with left-and right-movers interchanged, on the other side.

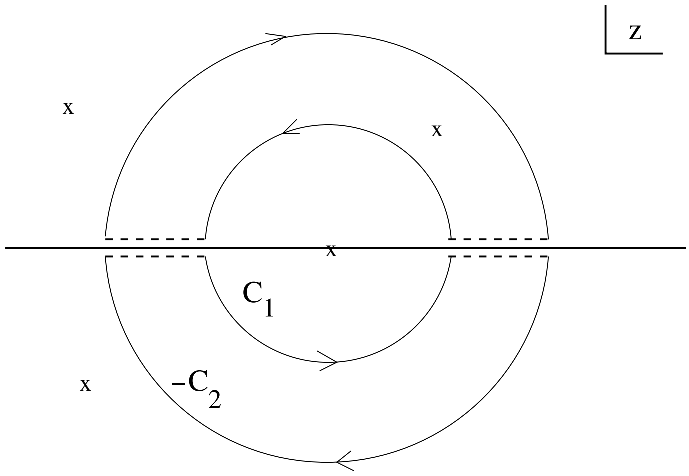

Figure 1: Conformal Ward identities are obtained by inserting $\oint_C [Tf(z)dz - \bar{T}f(\bar{z})d\bar{z}]$ in correlation functions. In deforming the contour from C_1 to C_2 we pick up contributions from the broken-line segments. These cancel out provided $T - \bar{T}$ is continuous. The crosses in the figure stand for local field insertions.

2. Free scalar field

In this section we discuss conformal 'permeable' walls for a single free scalar field ϕ. This is the simplest setting in which to illustrate the main ideas and calculation tricks, which we will then apply and extend to other contexts.

2.1 Gluing conditions

Consider a free massless scalar field in 1+1 dimensions, $\phi(x,t)$. We are interested in scale-invariant defects described by the 'gluing' conditions:

$$\begin{pmatrix} \partial_x \phi \\ \partial_t \phi \end{pmatrix}_{x=-0} = M \begin{pmatrix} \partial_x \phi \\ \partial_t \phi \end{pmatrix}_{x=+0} \tag{2.1}$$

where ± 0 denote points just to the left or right of the wall, which is located at $x = 0$, and M is a constant 2×2 matrix. Energy conservation requires that[3]

$$T_{xt} = T_{++} - T_{--} = \partial_x \phi \partial_t \phi \tag{2.2}$$

be continuous across the defect. Alternatively, notice that the conformal transformations which leave invariant the $x = 0$ worldline, are generated by the operators $[f(x^+)T_{++} - f(x^-)T_{--}]$. In the Wick-rotated theory, we can obtain the corresponding Ward identities by inserting a contour integral of these operators in correlation functions. Continuity of (2.2) ensures that one can deform the contour, so as to only pick contributions from field insertions. This is illustrated in figure 1.

[3]The light-cone coordinates are taken to be $x^{\pm} = t \pm x$, so that $\partial_{\pm} = \frac{1}{2}(\partial_t \pm \partial_x)$.

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The continuity of T_{xt} implies that M must be an element of $O(1,1)$. This group has four disconnected components,

$$M = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \text{or} \qquad M' = \pm \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \tag{2.3}$$

with λ a real positive number. We will 'compactify' the group by allowing also the singular values 0 and $\pm\infty$, so that λ runs over the entire compactified real line. As a result, the four disconnected components merge into two, which can be parametrized as follows:

$$M(\vartheta) \qquad \text{and} \qquad M'(\vartheta)\,, \qquad \text{with } \vartheta \equiv \arctan\lambda \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

We will see in the following subsections that this parametrization is natural.

The singular values of λ correspond to *perfectly reflecting* defects, for which the fields on either side don't communicate. Gluing derivatives with $M(0)$, for example, implies that $\partial_t\phi(+0) = \partial_x\phi(-0) = 0$. This is a standard Neumann condition for the field to the left of the wall, and a Dirichlet condition for the field on the right. Let us denote it by 'ND' (not to be confused with the mixed boundary conditions one often writes for the annulus). As can be, likewise, easily checked, $M(\pm\pi/2)$, $M'(0)$ and $M'(\pm\pi/2)$ correspond, respectively, to DN, NN and DD boundary conditions.

At the opposite extreme of the spectrum one has the four *perfectly transmitting* cases, corresponding to the special values $|\lambda| = 1$. Clearly, $M(\pi/4) = \mathbf{1}$ gives continuous derivatives — there is no defect in this special case. Gluing with $M(-\pi/4)$ makes ϕ jump to $-\phi$, but both left- and right-moving waves are still fully transmitted. The same is true for the two 'chiral defects' $M'(\pm\pi/4)$. For one of them left-moving waves are continuous across the wall, while right-moving waves pick a minus sign. For the other, the roles of left and right are reversed. If we were to let x be an angle coordinate, the four perfectly-transmitting walls would give rise to PP, AA, PA and AP boundary conditions for $(\partial_+\phi,\, \partial_-\phi)$.

The general defects interpolate between these standard cases. They are 'permeable', i.e. partially-reflecting and partially-transmitting. The two disconnected components of their moduli space are exhibited as two half-circles in figure 2. Sending $\lambda \to 1/\lambda$ exchanges, as can be easily seen, x- and t-derivatives on both sides. This is, therefore, the action of a T-duality transformation on the 'permeable defects.

2.2 S-matrix and Casimir energy

The defects in the first connected component of $O(1,1)$ have a simple realization as discontinuities in the radius of compactification of the scalar field. Indeed, let the field $\tilde\phi \equiv \tilde\phi + 2\pi$ be continuous in the entire plane, but have a discontinuous action

$$I = 2r_1^2 \int_{x<0} \partial_+\tilde\phi\partial_-\tilde\phi + 2r_2^2 \int_{x>0} \partial_+\tilde\phi\partial_-\tilde\phi\,. \tag{2.4}$$

Varying I gives the boundary conditions at $x = 0$:

$$r_1^2\partial_x\tilde\phi\Big|_{-0} = r_2^2\partial_x\tilde\phi\Big|_{+0}\,. \tag{2.5}$$

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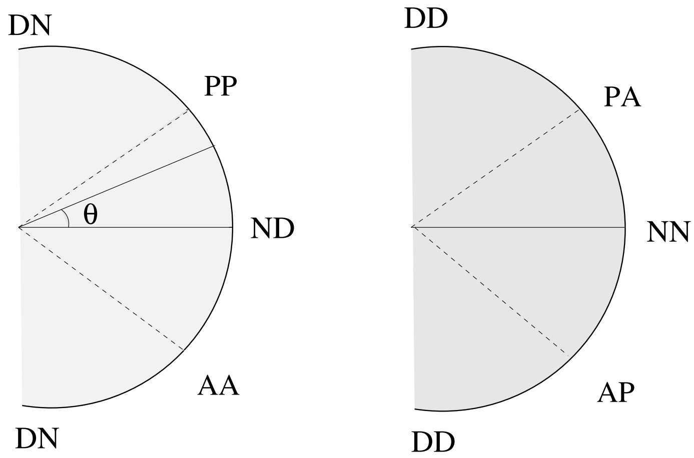

Figure 2: The moduli space of gluing matrices, $M(\vartheta)$ on the left and $M'(\vartheta)$ on the right, where $\vartheta = \arctan\lambda \in [-\pi/2, \pi/2]$. Perfectly-reflecting walls are labeled by the two boundary conditions, Dirichlet (D) or Neumann (N), on either side of the defect. Totally-transmitting defects are labeled by the periodicity properties of $(\partial_+\phi, \partial_-\phi)$ when x is compactified on a circle.

Redefining the scalar field so as to normalize its energy-momentum tensor,

$$\phi \equiv \begin{cases} r_1\tilde{\phi} & x < 0 \\ r_2\tilde{\phi} & x > 0\,, \end{cases}$$

leads precisely to the discontinuity equation (2.1), where the argument of the gluing matrix $M(\vartheta)$ obeys

$$\tan\vartheta = \lambda = \frac{r_2}{r_1}\,. \tag{2.6}$$

Thus, the parameter $\lambda = \tan\vartheta$ is related to the multiplicative discontinuity of the compactification radius across the wall. We will see the geometric significance of this fact in the following subsection.

Another useful characterization of the defects is in terms of a 'scattering matrix', from which one can read directly the reflection and transmission coefficients. Let us, for ease of notation, call ϕ^1 the field to the left of the wall, and ϕ^2 the field to the right. Then $\partial_-\phi^1$ and $\partial_+\phi^2$ can be expanded in terms of 'incoming waves', while $\partial_+\phi^1$ and $\partial_-\phi^2$ can be expanded in terms of 'outgoing waves' (as illustrated in figure 3). Strictly-speaking one cannot define asymptotic states for a massless 2d field, but this will not be important for our discussion here.

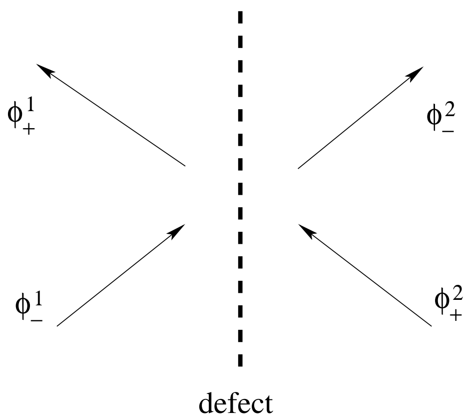

Figure 3: The incoming and outgoing waves can be related by the matrix S.

With the help of some linear algebra, we can write the gluing conditions (2.1) in the equivalent

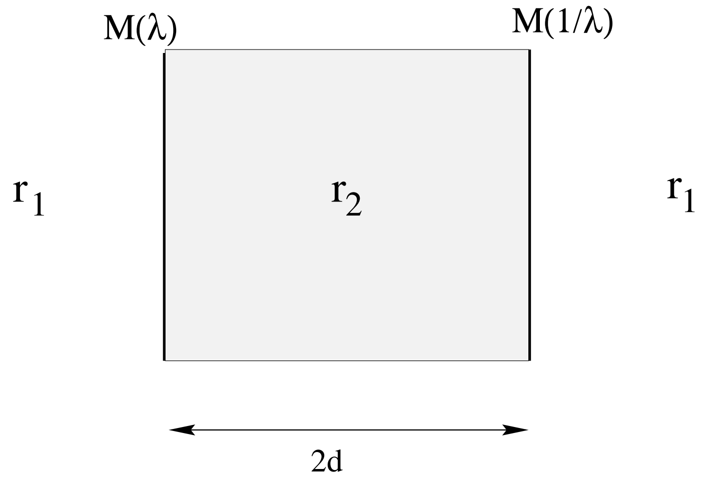

Figure 4: The region of rescaled radius ($r_2 = \lambda r_1$) bounded by a defect and an anti-defect. Time runs in the upward direction. The interfaces feel an attractive Casimir force.

form

$$\begin{pmatrix} \partial_- \phi^1 \\ \partial_+ \phi^2 \end{pmatrix} = S \begin{pmatrix} \partial_+ \phi^1 \\ \partial_- \phi^2 \end{pmatrix}, \tag{2.7}$$

where

$$S = \begin{pmatrix} \cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & -\cos 2\vartheta \end{pmatrix} \qquad \text{and} \qquad S' = \begin{pmatrix} \cos 2\vartheta & -\sin 2\vartheta \\ \sin 2\vartheta & \cos 2\vartheta \end{pmatrix}. \tag{2.8}$$

The orthogonal matrices S and S' relate incoming to outgoing waves at the defect. They are independent of the wave-frequency, as required by conformal invariance. Furthermore, they are off-diagonal for $\vartheta = \pm\pi/4$, corresponding to a perfectly-transmitting defect, and diagonal for ϑ a multiple of $\pi/2$, which corresponds to total reflection (see figure 2).

One simple observable, that can be expressed in terms of the scattering matrix, is the Casimir force between a defect and an anti-defect. Consider, to be specific, an interval inside which the radius of the scalar field jumps from r_1 to r_2,

$$I = \left(2r_1^2 \int_{-\infty}^{-d} + 2r_2^2 \int_{-d}^{d} + 2r_1^2 \int_{d}^{\infty} \right) \partial_- \tilde{\phi} \partial_+ \tilde{\phi}\,. \tag{2.9}$$

We assume that $\tilde{\phi}$ is continuous in the entire plane. It follows from our previous discussion, that there is a defect $M(\vartheta)$ located at $x = -d$, and an anti-defect with gluing matrix $M(\pi/2 - \vartheta)$ at $x = d$, where ϑ is given by equation (2.6). The setup is illustrated in figure 4.

In order to calculate the zero-point energy, we put the configuration in a larger box of size $2L$ so as to discretize the allowed frequencies. The presence of the defects in the middle induces a d-dependent shift in these frequencies, thereby modifying the zero-point sum. Taking $L \to \infty$ removes the dependence on the precise boundary conditions in the larger box, which can thus be chosen at will for convenience. What is left behind is a Casimir energy describing the interaction of the wall and antiwall. The calculation is rather subtle, because of the need to regularize the ultraviolet, and can be found in appendix A.

The result is

$$\mathcal{E} = -\frac{1}{8\pi d}\,\mathrm{Li}_2(\mathcal{R}^2)\,, \tag{2.10}$$

where $\mathrm{Li}_2(x) = \sum_1^\infty x^n/n^2$ is the dilogarithm function [23], and $\mathcal{R}$ is the reflection amplitude,

$$\mathcal{R} = \cos 2\vartheta = \frac{1-\lambda^2}{1+\lambda^2}\,. \tag{2.11}$$

For weak reflection the energy vanishes (as it should) quadratically:

$$\mathcal{E} \simeq -\frac{\mathcal{R}^2}{8\pi d} + o(\mathcal{R}^4)\,. \tag{2.12}$$

Total reflection, on the other hand, corresponds to $\mathcal{R} = \pm 1$. Since $\mathrm{Li}_2(1) = \pi^2/6$, one recovers the standard Casimir energy for a massless scalar field in a box in this special case.

2.3 Folding trick

The permeable defects of the previous sections can be described as regular D-branes, after 'folding' the plane along the defect line. This simple but powerful trick is well-known in the condensed-matter literature, and has been used for instance in the study of fracture lines for the Ising model [5]. To be more precise, let us define a 'conjugate' field in the left-half plane by mirror reflecting the field on the right,

$$\hat\phi^2(x,t) \equiv \phi^2(-x,t) \qquad \text{for } x \le 0\,. \tag{2.13}$$

The gluing conditions (2.1) with gluing matrix $M(\vartheta)$ read:

$$\partial_t(\cos\vartheta\,\phi^1 - \sin\vartheta\,\hat\phi^2)\Big|_0 = \partial_x(\sin\vartheta\,\phi^1 + \cos\vartheta\,\hat\phi^2)\Big|_0 = 0\,. \tag{2.14}$$

These are the boundary conditions for a D1-brane stretching along the direction ϑ in the $(\phi^1,\hat\phi^2)$ plane. The parametrization of the defects in terms of an angle variable can now be recognized as most natural. Note that bosonic D-branes are unoriented, which is why ϑ runs only over half a circle. Note also that the relation (2.6) between ϑ and the radii, in the case of periodically-identified fields, ensures that the D1-brane is compact. These facts are illustrated in figure 5.

To see the power of the folding trick, let us now rederive the Casimir energy of the previous subsection. We will need the conformal boundary state (see [24, 25] for nice reviews) that describes the D1-brane (2.14) in the closed-string language,

$$|\vartheta\,\rangle\rangle = \mathcal{N}\prod_{n=1}^{\infty}\exp\left(-\frac{1}{n}a^i_{-n}\tilde a^j_{-n}S_{ij}\right)|0;\varphi_\perp, w_{||}\rangle\,. \tag{2.15}$$

Here $a_n^{1,2}$ are the canonically-normalized left-moving oscillators for the fields ϕ^1 and $\hat\phi^2$, and $\tilde a_n^{1,2}$ are the corresponding right-moving oscillators. Note that these are the oscillators in the closed-string channel, where the roles of space and time are interchanged (we need of course to Wick rotate the coordinate t and to compactify it on a circle). The matrix

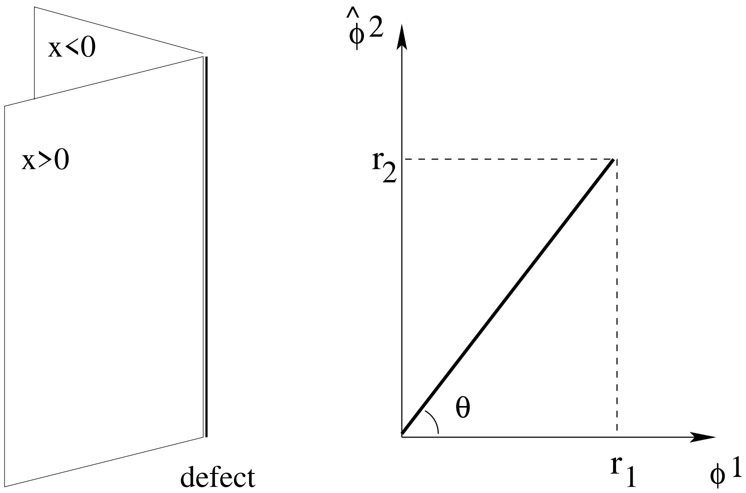

Figure 5: Folding the plane along the defect line leads to a description of the permeable defects as regular D-branes in a two-dimensional target space.

S is given by equation (2.8), and $\mathcal{N}$ is a normalization factor. Finally, $|0;\varphi_\perp, w_\|\rangle$ is the oscillator ground state, also characterized by the transverse position $\varphi_\perp$ of the D1-brane, and by the Wilson line $w_\|$ on its worldvolume. To simplify notation, we will suppress the dependence on these zero modes in what follows. Neither the normalization $\mathcal{N}$ nor the zero modes will, in any case, contribute to the Casimir energy that interests us here. The reader can verify easily that

$$(a_n^i + S_{ij}\tilde{a}^j_{-n})|\vartheta\rangle\rangle = 0\,, \tag{2.16}$$

which are the standard gluing conditions for the diagonal D-brane of figure 5 in the closed-string channel [24, 25].

In order to calculate the Casimir energy let us periodically identify $x \equiv x + 2L$. This differs from the Dirichlet conditions used in appendix A, but the difference will go away in the limit of infinite L. We also let the time coordinate have period T. The vacuum energy for the configuration of figure 4 can be written as follows in the closed-string channel:

$$\mathcal{E} = \lim_{T\to\infty} -\frac{1}{T}\log\langle\langle\vartheta|e^{-H^1 4\pi(L-d)/T}e^{-H^2 4\pi d/T}|\vartheta\rangle\rangle\,, \tag{2.17}$$

with H^1 and H^2 the free-field hamiltonians of ϕ^1 and $\hat{\phi}^2$. The limit $L\to\infty$ projects onto the ground state of ϕ^1, so that only the $\hat{\phi}^2$ oscillators should be kept in the expression (2.15) for the boundary state. The above matrix element thus becomes

$$\langle 0|\prod_{n=1}^{\infty}\exp\left(-\frac{1}{n}a_n^2\tilde{a}_n^2\cos 2\vartheta\right)e^{-H^2 4\pi d/T}\prod_{n=1}^{\infty}\exp\left(-\frac{1}{n}a_{-n}^2\tilde{a}_{-n}^2\cos 2\vartheta\right)|0\rangle =$$

$$= \mathcal{N}^2\prod_{n=1}^{\infty}\left(1-\cos^2 2\vartheta\; e^{-n8\pi d/T}\right)^{-1}\,. \tag{2.18}$$

Taking the logarithm converts the product into a sum, which in the limit $T \to \infty$ reduces to a continuous integral,

$$\mathcal{E} = \frac{1}{8\pi d} \int_0^1 \frac{dy}{y} \log(1 - y \cos^2 2\vartheta) \,. \tag{2.19}$$

Using, finally, the integral representation of the dilogarithm function [23],

$$\int_O^z \frac{dw}{w} \log(1 - w) = -\,\mathrm{Li}_2(z) \,, \tag{2.20}$$

we recover precisely the result (2.10) of appendix A. The dilogarithm function has appeared before in the CFT literature [26], but the present context is, in our opinion, particularly simple. The expression (2.18) for the matrix element has also appeared in the literature before, under the name 'quantum dilogarithm' [27, 28].

We can also evaluate (2.17) for L finite. If we denote $q_1 = \exp(-4\pi d/T)$ and $q_2 = \exp(-4\pi(L-d)/T)$, then the relevant matrix element reads:

$$\mathcal{N}^2 \prod_{n=1}^{\infty} \left[1 - (q_1^{2n} + q_2^{2n}) \cos^2 2\vartheta - 2 q_1^n q_2^n \sin^2 2\vartheta + q_1^{2n} q_2^{2n}\right]^{-1} . \tag{2.21}$$

Sending $q_2 \to 0$ gives back the expression (2.18) as expected. When $d = L/2$, on the other hand, the matrix element reduces to $\mathcal{N}^2 \prod_{n=1}^{\infty}(1 - q^{2n})^{-2}$, where $q = q_1 = q_2$. The Casimir energy is independent of ϑ in this special case. This is consistent with the fact that the mass subtraction for a closed string (corresponding to $\vartheta = \pi/4$) is twice the subtraction for an open string (which corresponds to $\vartheta = 0$ or $\pi/2$).

We conclude this section with a brief discussion of other gluing conditions, and in particular those corresponding to the matrices $M'(\vartheta)$. Let ${}^*\phi^2$ be the field T-dual to $\hat{\phi}^2$, which obeys $\partial_t {}^*\phi^2 = \partial_x \hat{\phi}^2$ and $\partial_x {}^*\phi^2 = \partial_t \hat{\phi}^2$. It follows from the relation

$$M'(\vartheta) = M(\vartheta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{2.22}$$

that the M' gluing condition describes a D1-brane in the direction ϑ on the $(\phi^1,\ {}^*\phi^2)$ plane. The T-duality that takes us back to the $(\phi^1, \hat{\phi}^2)$ plane, transforms this D1-brane into a D2-brane with a non-vanishing worldvolume magnetic flux [29]. In the simplest case of a compact scalar and a diagonal D1-brane, as in figure 5, the T-dual configuration is characterized by one unit of magnetic flux. We should stress, however, that the relation (2.6) between the angle ϑ and the radii is consistent, but by no means unique. It was derived from the hypothesis that the field $\tilde{\phi}$ of section 2.2 should be continuous across the wall. A more general consistent hypothesis is that $\tilde{\phi}(-0) = n\tilde{\phi}(+0)$, leading to the relation

$$\tan\vartheta = \lambda = \frac{r_2}{n r_1} \,. \tag{2.23}$$

This corresponds (after folding) to a D1-brane that winds n times around dimension 1, but only a single time around dimension 2. The T-dual configuration is a D2-brane carrying n units of magnetic flux. As will become in fact clear in the following section, *any* consistent D-brane configuration on the two-torus can be 'unfolded' to a conformally-invariant interface of the one-scalar theory.

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3. Supersymmetry and generalizations

The analysis of the previous section can be extended in several directions. One may consider abstract gluings of conformal theories, mutliple interfaces or junctions, Calabi-Yau sigma models, or orbifold theories. Another important question concerns the supersymmetry properties of the walls. In this section we will elaborate on some of these various issues.

3.1 Fermions and supersymmetry

The $N = (1,1)$ supersymmetric extension of the free-scalar model has a pair (ψ_+, ψ_-) of Weyl-Majorana fermions, which are the superpartners of the field ϕ. Conformal invariance requires continuity of $(T_{++} - T_{--})$ for the fermions. Supersymmetry, on the other hand, further requires that

$$(G_+ + \eta G_-)|_{-0} = \pm(G_+ \pm \eta G_-)|_{+0}\,, \tag{3.1}$$

where $G_\pm$ are the left and right supercurrents, and $\eta = \pm 1$. For a single wall, the three sign ambiguities in this condition can be absorbed in redefinitions of the fermion fields. The signs involving only the fields on the same side of the wall are basically irrelevant (except possibly if x is compactified) and we will henceforth take them to be positive. The third sign, η, on the other hand, involves fields on both sides of the wall, and will therefore be important when two or more interfaces are present. As we will see, η distinguishes an interface from an anti-interface.

In order to make the supersymmetry manifest, we will show how these boundary conditions arise directly in superspace. Consider the general $N = (1,1)$ supersymmetric sigma model with action

$$I = \int dx\, dt\, d^2\theta\, [G_{IJ}(\Phi) + B_{IJ}(\Phi)]\, D_+\Phi^I D_-\Phi^J\,, \tag{3.2}$$

where

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\pm\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right). \tag{3.3}$$

If this sigma model is the CFT on the left of the domain wall, we need to find the variation of the action, and match it to the corresponding variation on the right side of the wall. We assume here, as we did until now, that the domain wall does not support any independent degrees of freedom.

The variation of the action (3.2) yields the following boundary term:

$$\begin{aligned}\delta I = -\int dt\Big[&\frac{1}{2}\Sigma_J\delta(D_+\Phi^J + D_-\Phi^J) + \frac{1}{2}(D_+ + D_-)\Phi^J\delta\Sigma_J - \\ &- \delta\Phi^J(D_+ + D_-)\Sigma_J\Big]_{x=-0,\theta^\pm=0}\,,\end{aligned} \tag{3.4}$$

where

$$\Sigma_J = [G_{JK} + B_{JK}]D_+\Phi^K - [G_{JK} + B_{JK}]D_-\Phi^K\,. \tag{3.5}$$

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In deriving equation (3.4) we used the equation of motion for the auxiliary field, which is the top component of the superfield Φ. In addition, we have dropped the variation of a pure boundary term, $-\delta I_b$, with

$$I_b = \frac{1}{2}\int dt\, B_{IJ}(\Phi)\left(D_+\Phi^I D_+\Phi^J + D_-\Phi^I D_-\Phi^J\right). \tag{3.6}$$

This is of course absent for $B_{IJ}=0$, and in particular if there is only one superfield. More generally, we should have included this boundary term in the action (3.2) in order to arrive at the above variation.

The form of the variation (3.4) suggests that we introduce two 'half' superfields [30] as follows:[4]

$$\tilde{\Phi}^J(x,t,\theta) = \Phi^J\Big|_{\theta^+=\theta^-=\theta} \qquad \text{and} \qquad \tilde{\Sigma}_J(x,t,\theta) = \Sigma_J\Big|_{\theta^+=\theta^-=\theta}. \tag{3.7}$$

For example, in flat space and for zero B_{IJ} we find:

$$\tilde{\Phi}^J = \phi^J + \theta(\psi^J_+ + \psi^J_-) \qquad \text{and} \qquad \tilde{\Sigma}_J = (\psi^J_+ - \psi^J_-) + 2\theta\partial_x\phi^J\,. \tag{3.8}$$

One can now verify easily that, if these half superfields are continuous across the wall, then the variations (3.4) of the left and right CFTs will precisely cancel out each other. In addition, a manifest $N=1$ supersymmetry will be preserved, since everything can be expressed in terms of half superfields. The superderivate in this half superspace is defined as:

$$D \equiv D_+ + D_- = \partial_\theta + 2\theta\partial_t\,, \tag{3.9}$$

and since it does not contain a derivative of x, it acts indeed along the interface. Another way of arriving at the above conclusion, is by constructing the superfield combination

$$\Theta \equiv \frac{1}{8}\left[D^2\tilde{\Phi}^J\tilde{\Sigma}_J + D\tilde{\Phi}^J D\tilde{\Sigma}_J\right] = G_+ + G_- + \theta(T_{++} - T_{--})\,. \tag{3.10}$$

From this one sees immediately that continuity of the half superfields (3.7) across the wall implies, indeed, the boundary conditions given in (3.1), with η (and all the other signs) chosen to be positive.

The choice $\eta=-1$ corresponds to another set of half-superfields, which are obtained by setting $\theta^+ = -\theta^- = \theta$. The combination (3.10), with $D \equiv D_+ - D_-$, has now $G_+ - G_-$ as its lowest component (and the same upper component as above). Continuity of this new set of half-superfields respects, therefore, the $\eta=-1$ superconformal-invariance conditions. Two interfaces with opposite values of η break completely all the supersymmetry.

If there are more than one superfield, the vanishing of (3.4) is guaranteed by the more general boundary conditions

$$\begin{pmatrix}\tilde{\Sigma}_J \\ D\tilde{\Phi}^J\end{pmatrix}_{x=-0} = M\begin{pmatrix}\tilde{\Sigma}_J \\ D\tilde{\Phi}^J\end{pmatrix}_{x=+0}, \tag{3.11}$$

[4] See also [31] for a recent detailed analysis of supersymmetry-preserving boundary conditions in general $N=(1,1)$ sigma-models.

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with the constant matrix $M \in O(d,d)$. Of course, our discussion here is entirely classical, and superconformal symmetry could be broken by quantum corrections. Furthermore, one needs to check compatibility of the above conditions with the global structure of the target space of the sigma model. Thus, in general, only a limited subset of $O(d,d)$ gluings will be allowed.

The calculation of the Casimir energy of the previous section can be extended easily to the superconformal case. The gluing conditions for the fermionic fields that supersymmetrize equation (3.11) are:

$$\begin{pmatrix} \psi_-^1 \\ \psi_+^2 \end{pmatrix} = S(\eta) \begin{pmatrix} \psi_+^1 \\ \psi_-^2 \end{pmatrix}, \tag{3.12}$$

where

$$S(\eta) = \begin{pmatrix} \eta \cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & -\eta \cos 2\vartheta \end{pmatrix}, \tag{3.13}$$

with a similar expression for S'. The factors of η in the gluing matrix follow from the fact that changing η is the same as flipping the sign of the ψ_-^j. The fermionic part of the boundary state that imposes these gluing conditions is

$$|\vartheta, \eta\rangle\rangle_F = \mathcal{N}' \prod_{r>0} \exp\left(i\psi_{-r}^i \tilde{\psi}_{-r}^j S_{ij}(\eta) \right) |0\rangle . \tag{3.14}$$

The factor of i in the exponent arises in going from the open to the closed-string channel [25], and $\mathcal{N}'$ is a (irrelevant for us) normalization. The frequencies r can be either integer or half-integer, depending on whether we are in the Ramond or Neveu-Schwarz sector of the closed-string.

Proceeding as in section 2.3 we obtain the following expression for the Casimir energy:

$$\mathcal{E} = \lim_{T\to\infty} -\frac{1}{T} \log \frac{\prod_r (1 - \eta_L \eta_R \cos^2 2\theta e^{-r8\pi d/T})}{\prod_n (1 - \cos^2 2\theta e^{-n8\pi d/T})} . \tag{3.15}$$

Since $T \to \infty$, the result does not depend on the choice of integer or half-integer r. What does make a difference is whether the left and right interfaces are of the same or of opposite type: $\eta_L \eta_R = +1$ or -1. In the first case supersymmetry is preserved and the Casimir energy is zero. In the second case one finds that

$$\mathcal{E} = -\frac{1}{8\pi d} \left[\mathrm{Li}_2(\mathcal{R}^2) - \mathrm{Li}_2(-\mathcal{R}^2) \right] = -\frac{1}{8\pi d} \left[2\,\mathrm{Li}_2(\mathcal{R}^2) - \frac{1}{2}\,\mathrm{Li}_2(\mathcal{R}^4) \right] . \tag{3.16}$$

The last equality follows from a standard dilogarithmic identity. Writing $\mathcal{E}$ in this form shows that in the case of total reflection ($\mathcal{R} = \pm 1$) the result is $3/2$ times the bosonic contribution. This is indeed the vacuum energy of a superfield with conventional Neveu-Schwarz boundary conditions. For weak reflection, on the other hand, the bosonic and fermionic contributions to the Casimir energy are equal.

Let us finally discuss $N = (2,2)$ sigma models with a target space that is a Kähler manifold. In this case, the sigma-model action takes the form

$$I = \int dx\, dt\, d^4\theta K(\Phi^i, \bar{\Phi}^{\bar{i}}) , \tag{3.17}$$

where $\Phi, \bar\Phi$ are (anti)chiral superfields that satisfy $\bar D_\pm \Phi^i = D_\pm \bar\Phi^{\bar i} = 0$. The superderivatives are:

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + 2\bar\theta^\pm \partial_\pm \, , \qquad \bar D_\pm = \frac{\partial}{\partial\bar\theta^\pm} + 2\theta^\pm\partial_\pm \, . \tag{3.18}$$

Repeating our previous analysis, we find that the variation of the action can be written again in terms of half superfields. The relevant half superfields now are:

$$\begin{aligned}
\varphi^i(x,t,\theta,\bar\theta) &= \Phi^i\Big|_{\theta^+=\theta^-=\theta,\, \bar\theta^+=\bar\theta^-=\bar\theta}\,, \\
\bar\varphi^{\bar i}(x,t,\theta,\bar\theta) &= \bar\Phi^{\bar i}\Big|_{\theta^+=\theta^-=\theta,\, \bar\theta^+=\bar\theta^-=\bar\theta}\,, \\
\Lambda_i(x,t,\theta,\bar\theta) &= \partial_i\partial_{\bar j}K(\bar D_+ - \bar D_-)\bar\Phi^{\bar j}\Big|_{\theta^+=\theta^-=\theta,\, \bar\theta^+=\bar\theta^-=\bar\theta}\,, \\
\bar\Lambda_{\bar i}(x,t,\theta,\bar\theta) &= \partial_i\partial_{\bar j}K(D_+ - D_-)\Phi^i\Big|_{\theta^+=\theta^-=\theta,\, \bar\theta^+=\bar\theta^-=\bar\theta}\,.
\end{aligned} \tag{3.19}$$

The half-superspace coordinates are θ and $\bar\theta$, with derivatives $D = (\partial_\theta + 2\bar\theta\partial_t)$ and $\bar D = (\partial_{\bar\theta} + 2\theta\partial_t)$. By requiring the above half superfields to be continuous accross the domain wall, we automatically preserve one $N = 2$ algebra. The generators of this algebra are the components of the half superfield

$$(D\varphi^i)\Lambda_i + \bar\Lambda_{\bar i}\bar D\bar\varphi^{\bar i}\,. \tag{3.20}$$

These are clearly continuous across the wall, once the fields in (3.19) are themselves continuous. Note that the lowest component of (3.20) is the difference of the left- and right-moving U(1) currents.

More generally, if the target space is d-complex-dimensional, there is an $O(d,d,\mathbb{C})$ family of candidate boundary conditions. The subgroup $\mathrm{GL}(d,\mathbb{C}) \subset O(d,d,\mathbb{C})$ has a simple interpretation in terms of holomorphic branes in $\mathcal{M}_L \times \mathcal{M}_R$, where $\mathcal{M}_{L,R}$ are the two target manifolds on either side of the interface. Indeed, let v^i be complex coordinates for $\mathcal{M}_L$ and w^i complex coordinates for $\mathcal{M}_R$. Then $v^i = A^i_j w^j$ defines, in a local patch, a holomorphic d-complex dimensional brane. When this brane can be defined globally (we will discuss such an example in the following subsection) then it gives rise to a $N = 2$ superconformal interface. Since holomorphic branes are BPS, we expect them to survive in the quantum theory, at least in the large-volume limit.

3.2 Generalizations

The folding trick allows us to discuss conformal-field-theory gluings more abstractly. Start with the tensor product of two conformal theories, $\mathrm{CFT}_1 \otimes \mathrm{CFT}_2$, defined on the euclidean half plane, $\mathrm{Im}\, z \geq 0$. The two theories need not be identical, nor even have equal central charges. Conformal boundary conditions are described by a boundary state $|\mathcal{B}\rangle\rangle$, which satisfies

$$\left(L_n^{(1)} + L_n^{(2)} - \bar L_{-n}^{(1)} - \bar L_{-n}^{(2)}\right)|\mathcal{B}\rangle\rangle = 0\,. \tag{3.21}$$

Here $L_n^{(1)}$ and $\bar L_n^{(1)}$ are the left-moving, respectively right-moving Virasoro generators of CFT_1, in the closed-string channel, and similarly for CFT_2 (we drop the tildes for ease of notation). If we 'unfold' CFT_2 unto the lower half-plane, $\mathrm{Im}\, z \leq 0$, the roles of its holomorphic and antiholomorphic fields are interchanged. Condition (3.21) then precisely ensures the continuity of $T_{zz} - \bar{T}_{\bar{z}\bar{z}}$ on the real axis. In this way, any conformal boundary state in the tensor-product theory can be unfolded into a conformal interface, and vice-versa.

A trivial situation arises whenever the boundary state can be factorized,

$$|\mathcal{B}\rangle\rangle_{\mathrm{reflect}} = |\mathcal{B}_1\rangle\rangle \otimes |\mathcal{B}_2\rangle\rangle \, . \tag{3.22}$$

In this case $L_n - \bar{L}_{-n}$ vanish for each theory separately, so that T_{xt} is zero at the interface. There can, therefore, be no transfer of energy across the wall, and the two conformal field theories are decoupled.[5] At the opposite end of the spectrum are the perfectly-transmitting defects, for which

$$\left(L_n^{(1)} - \bar{L}_{-n}^{(2)}\right) |\mathcal{B}\rangle\rangle_{\mathrm{transmit}} = \left(L_n^{(2)} - \bar{L}_{-n}^{(1)}\right) |\mathcal{B}\rangle\rangle_{\mathrm{transmit}} = 0 \, . \tag{3.23}$$

Such states obviously exist when the two CFTs are identical, but not only. For instance, for the scalar field of section 2 one may consider a D1 brane at 45^o, even if the radii on the two sides of the interface are not the same. Generic permeable defects are those for which the boundary state $|\mathcal{B}\rangle\rangle$ is of neither of the above two special types.

As well-known, the Virasoro gluing equations (3.21) must be supplemented, in general, by global consistency conditions (for reviews and references see [32, 33, 34]). For instance, the annulus diagram must be a partition function with integer multiplicities in the open channel [1]. Such conditions should be obeyed automatically by defects described by a local action principle, like those we have considered up to now. From a more algebraic point of view, it should be sufficient to ensure that the state $|\mathcal{B}\rangle\rangle$ in the tensor-product theory is consistent. The consistency of the bulk and boundary operator algebra can be, indeed, verified before the procedure of 'unfolding'. The boundary operators, that are consistent with the sewing constraints in the tensor theory, will 'unfold' into local operators that live on the interface.

These considerations can be generalized easily to any number of adjacent parallel defects. One must fold along the interfaces repeatedly, as illustrated in figure 6a, so as to make an annulus with many sheets. The boundary conditions at the folds are boundary states of the product theory $\mathrm{CFT}_1 \otimes \mathrm{CFT}_2 \otimes \cdots \mathrm{CFT}_k$, where CFT_m is the theory on the mth sheet and for even m the left- and the right-movers must be exchanged. Note that one can introduce extra folds with purely-transmitting boundary conditions. One can also consider multiple junctions of CFTs, as illustrated in figure 6b (for an earlier study of field theories on string junctions see [35]). Extending the calculations of the previous section in such contexts is a straightforward exercise that we do not pursue.

The construction of permeable interfaces of strongly-interacting CFTs is a very interesting question, to which we hope to return in future work. Here, we want to conclude

[5] Any linear combination of states of type (3.22) will, likewise, give perfect reflection. By an abuse of language, we keep refering to such states as 'factorizable', since the two conformal theories dont talk, except possibly via correlated boundary conditions.

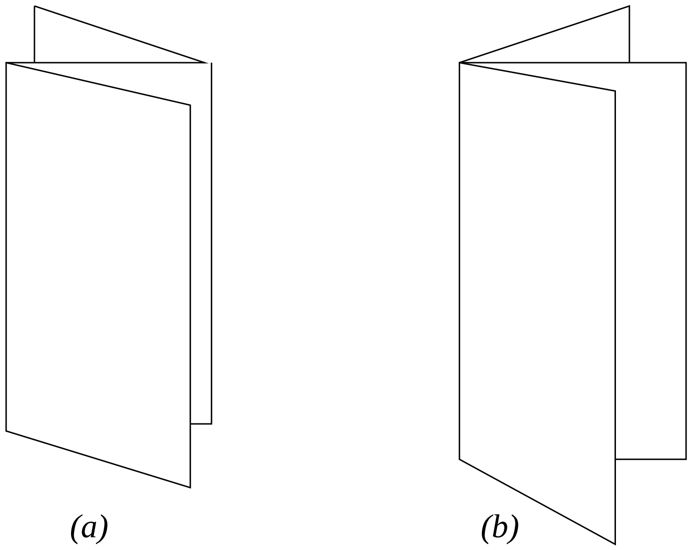

Figure 6: The folding of (a) two neighbouring interfaces, and (b) a triple junction of conformal theories.

our discussion with a few more simple examples of domain walls. First, let us consider the case of several free scalar fields, n_1 on the left and n_2 on the right of the interface. The boundary states are (combinations) of planar branes in $n_1 + n_2$ dimensions, which are generically at angles and can carry a magnetic flux. If the scalar fields have canonically normalized stress tensors, the gluing conditions will be of the same form as (2.7), with S an orthogonal matrix that we write in terms of $n_i \times n_j$ blocks:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}. \tag{3.24}$$

Repeating the Casimir-energy calculation of section 2 gives:

$$\mathcal{E} = -\frac{1}{8\pi d}\,\mathrm{Tr}\left[\mathrm{Li}_2(S_{22}^{\;2})\right]. \tag{3.25}$$

Notice that the pressure on the walls only depends, as should be expected, on the reflection amplitudes of the conformal theory CFT_2 that lives in the space in between these walls.

For a less trivial example, let us discuss orbifolds. Consider the case where on either side of the interface lives a $c = 1$ orbifold theory, so that the tensor product CFT has target space $S^1/Z_2 \times S^1/Z_2$. A D1 brane winding once around each of the two covering circles has the generic form shown in figure 7. It is an inscribed parallelogram, with sides parallel to the two diagonals of the target space. There is, furthermore, a four-fold Chan-Paton multiplicity, corresponding to the four images of the D-brane in the covering torus. Marginal deformations change the shape of the parallelogram, while keeping its four angles fixed, and also turn on a Wilson line. At special point(s) of this moduli space, where the parallelogram collapses along a diagonal of target space, as in figure 7, the D1-brane decomposes into two, more elementary, fractional D-branes [36, 37, 38]. These are the basic branes of the tensor-product theory which, in the limit of equal radii ($r_1 = r_2$), unfold into perfectly-transmitting interfaces.

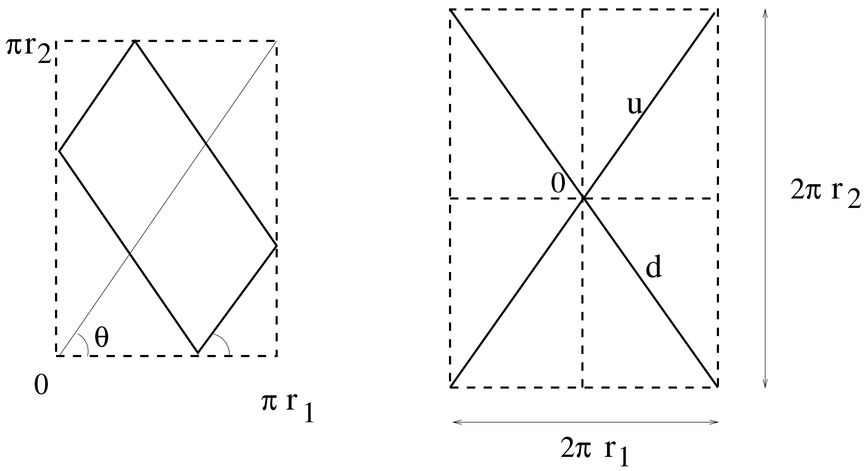

Figure 7: A regular D1-brane of the $S^1/Z_2 \times S^1/Z_2$ orbifold theory that winds once around each of the covering circles (left). When forced to go through the origin, this D-brane has a single, rather than three, images under reflections (right). In this case it can decompose (assuming also vanishing Wilson line) into two, more elementary, fractional branes.

We can extend the above discussion to $N = (2, 2)$ supersymmetric sigma models on orbifold spaces, like T^4/Z_2 or T^6/Z_3. Consider the latter example which is a (singular) Calabi-Yau surface with a unique complex structure and 36 Kähler moduli. Varying the nine untwisted moduli separately, for the two sigma models of the tensor product, will lead to diagonal branes that describe permeable interfaces. Vaying the 27 twisted moduli will blow up some of the orbifold fixed points. Since the complex structure is here unique, we expect the middle-dimensional holomorphic branes described in section 3.1 to survive.

4. The NS5/F1 system and holography

We will now apply the ideas of CFT domain walls to branes in AdS_3. Of special interest to us are the static one-branes extending all the way to spatial infinity [9]. Since these are codimension-one in the bulk, they separate two different supergravity vacua, distinguished by their charges. Correspondingly on the boundary we find $0 + 1$ dimensional domain walls separating two different CFTs, that a priori can have different central charges. Since the stable one-branes are supersymmetric and have AdS_2 geometry [9], the corresponding domain walls should be superconformal.

One way of trying to test this correspondence is by comparing the Casimir energy of the walls, both from the supergravity and from the CFT viewpoints. This is the two-dimensional analogue of the Wilson loop calculation [18, 19] in four dimensions. It is also one version of the more general Karch-Randall setup [16] in which two n-dimensional CFT's are glued together with a $(n - 1)$-dimensional CFT.

4.1 String theory setup

Our starting point is the type-IIB string compactification on a four-manifold M^4, which is either a four-torus or a K3 surface. The resulting six-dimensional theory contains a variety of strings. Besides the fundamental string and D-string of the uncompactified IIB theory, there are also D3 branes wrapping the various two cycles of M^4, as well as D5 and NS5 branes wrapping the entire manifold. The strings are labeled by a charge vector $\vec{q}$ in the lattice $\Gamma^{5,5+n}$, where $n = 0$ for $M^4 = T^4$ and $n = 16$ for $M^4 = K3$. Furthermore, there is a $O(5, 5+n, \mathbb{Z})$ duality group which permutes the different charges, keeping the invariant length $\vec{q}^{\,2}$ fixed. The moduli space of this string compactification is

$$O(5, 5+n, \mathbb{Z})\backslash O(5, 5+n)/O(5) \times O(5+n)\,. \tag{4.1}$$

We can distinguish two classes of BPS strings. First, those with a (primitive) charge vector of zero length, $\vec{q}^{\,2} = 0$, which lie in the U-duality orbit of the fundamental string. Such objects are weakly coupled in some corner of the moduli space, and can be chosen as the fundamental quanta in a perturbative expansion. Secondly, there are strings with $\vec{q}^{\,2}$ positive.[6] These can be always mapped, by a U-duality transformation, to a bound state of Q_1 fundamental strings and Q_5 NS fivebranes, where

$$\vec{q}^{2} = 2Q_1Q_5\,. \tag{4.2}$$

If the charge vector $\vec{q}$ is furthermore primitive, Q_1 and Q_5 are relatively prime and we have a well-defined bound state. We want to study the near-horizon decoupling limit for such a configuration. The relevant geometry is $\mathrm{AdS}_3 \times S_3 \times M^4$, and the dual supersymmetric CFT has total central charge $6N = 6\; Q_1Q_5$.

Picking a particular charge vector $\vec{q}$, reduces the duality group and moduli space. The remaining U-dualities, that are realized as T-dualities in the CFT, are given by the 'little group' $O(4, 5+n, \mathbb{Z})$ that preserves the charge vector $\vec{q}$. By the attractor mechanism [39, 40] some of the scalar fields that parametrize the moduli space take specific fixed values in the near-horizon region. More explicitly, if we use the Narain decomposition $\vec{q} = \vec{q}_L + \vec{q}_R$ with $\vec{q}^2 = \vec{q}_L^2 - \vec{q}_R^2$, the attractor equation gives[7] $\vec{q}_R = 0$. The moduli space of the supergravity solution is then reduced to the homogeneous space

$$O(4, 5+n, \mathbb{Z})\backslash O(4, 5+n)/O(4) \times O(5+n)\,. \tag{4.3}$$

Note that $|\vec{q}_L|$ is the tension of the background string. Note also that the full parameter space of the dual (spacetime) CFT includes many copies of the 'fundamental domain' (4.3), and has an intricate global structure [41].

This six-dimensional theory contains various string junctions where a string with charge $\vec{q}_1$ absorbs a string with charge $\vec{q}_2$ to form a string with charge $\vec{q}_1 + \vec{q}_2$. The superconformal walls are holographic duals of such junctions. We will choose a duality frame where the background $\vec{q}_1$ is built out of only fundamental strings and NS fivebranes. Its near-horizon

[6]For $M^4 = T^4$ the negative $\vec{q}^{\,2}$ strings are also supersymmetric.

[7]There are two natural bases for the charge vector: one with integer entries (counting different branes), and one giving the couplings to normalized 6d gauge fields. Our left-right decomposition uses the latter basis, which depends on the asymptotic values of the moduli in flat space. In a string junction the charge vector is of course conserved in either basis, but not after one imposes the attractor conditions $\vec{q}_R = 0$, since these may fix the moduli differently near the horizon of the individual strings.

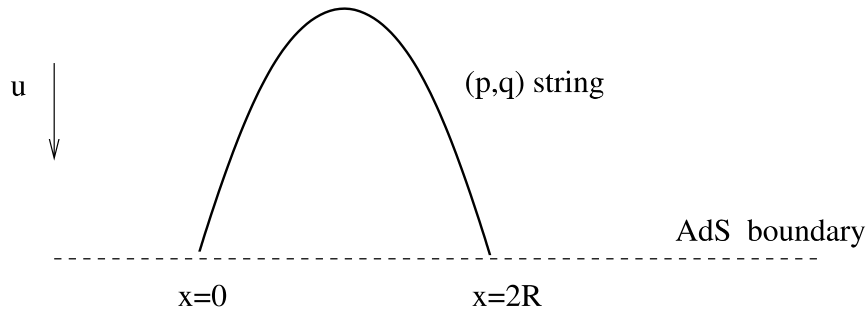

Figure 8: Stretched string between a wall and an antiwall.

geometry carries, therefore, Neveu-Schwarz fluxes only. The full type-IIB string theory can be described in this case by a Wess-Zumino-Witten model on the group manifold $\mathrm{SL}(2,\mathbb{R}) \times SU(2)$, together with a sigma model with M^4 target space. The Q_1-dependence appears through the six-dimensional string coupling, which is fixed by the attractor mechanism to be

$$\frac{1}{g_6^2} = \frac{Q_1}{Q_5}\,. \tag{4.4}$$

For a reliable supergravity approximation one needs therefore $Q_1 \gg Q_5 \gg 1$.

Let us consider now a second string with charge vector $\vec{q}_2$, stretching between two points, $x = 0$ and $x = 2R$, on the AdS_3 boundary as in figure 8. In the dual holographic field theory the string endpoints are a wall and an antiwall, separating two different CFTs. With the use of T-dualities we can map this second string to a configuration that does not contain D3-branes. Although we will mostly work with (p,q) strings below, the most general configuration can also involve D5- and NS5-branes. The U-dualities that preserve the vector $\vec{q}_1$ are, in general, insufficient to always map $\vec{q}_2$ to only fundamental strings and D-strings.

A (p,q) string like the one of figure 8 will only equilibrate if one applies a force to keep its two endpoints from collapsing. From the holographic point of view, this force is the Casimir attraction of the walls. We will now compute it in the supergravity approximation. In order to do a reliable calculation we assume that the tension $T_{(p,q)}$ of the probe string is much smaller than the tension $T(\vec{q}_1)$ of the background string, so that backreaction can be consistently neglected.

The calculation is similar in spirit to the Wilson-loop calculation in the supergravity limit of $N = 4$ super-Yang-Mills [18, 19]. The string coupling to the background B-field introduces, however, a new parameter at the technical level.

4.2 Supergravity calculation

The metric and B-field backgrounds of the $\mathrm{SL}(2,\mathbb{R})$ WZW model in Poincaré coordinates are

$$ds^2 = L^2\left[\frac{du^2}{u^2} + u^2(dx^2 - dt^2)\right] \qquad \text{and} \qquad B = L^2 u^2\, dx \wedge dt\,. \tag{4.5}$$

We denote, for short, by T and ρ the tension and NS charge density of the (p,q) string. Its energy, as measured by an observer sitting at radial position $u=1$, takes the form [9]

$$\mathcal{E} = 2L\int_0^R dx\,\left[T\sqrt{u^4+u'^{\,2}}-\rho u^2\right], \tag{4.6}$$

with $u' = du/dx$. Extremizing leads to the constant of motion

$$\left[\frac{Tu^4}{\sqrt{u^4+u'^{\,2}}}-\rho u^2\right] \equiv \rho C\,. \tag{4.7}$$

Setting $C=0$ corresponds to free boundary conditions at the endpoints. The string falls, in this case, towards the Poincaré horizon and never comes back. Its worldsheet has AdS_2 geometry. More generally, C and R are related implicitely by

$$R = \int_0^R dx = \int_\infty^{u_0}\frac{du}{u'}\,, \tag{4.8}$$

where u_0 is the minimum value of u, corresponding to $u'=0$. Solving (4.7) for u', and making the change of variables $w\equiv 1/u^2$, gives

$$R = \frac{1}{2}\int_0^{w_0}\frac{dw}{\sqrt{w}}\frac{Cw+\rho}{\sqrt{(T+\rho+Cw)(T-\rho-Cw)}}\,, \tag{4.9}$$

with $w_0 = \frac{T-\rho}{C}$. Performing the integrations we find

$$\sqrt{C} = \frac{\sqrt{2T}}{R}\left(\mathbf{E}(k)-\frac{1}{2}\mathbf{K}(k)\right)\,, \tag{4.10}$$

where $\mathbf{E}$ and $\mathbf{K}$ are the complete elliptic integrals,

$$\mathbf{E}(k) = \int_0^{\frac{\pi}{2}} da\,\sqrt{1-k^2\sin^2 a}\,, \qquad \text{and} \qquad \mathbf{K}(k) = \int_0^{\frac{\pi}{2}}\frac{da}{\sqrt{1-k^2\sin^2 a}}\,. \tag{4.11}$$

The argument of these integrals is a function of the tension and the NS-charge density of the probe string,

$$k^2 = \frac{T-\rho}{2T}\,. \tag{4.12}$$

Equation (4.10) expresses the integration constant C in terms of the separation of the string endpoints.

Let us next evaluate the energy. Substituting u' in equation (4.6) and changing again variables to $w=1/u^2$, leads to the expression:

$$\mathcal{E} = L\int_{\epsilon^2}^{w_0}\frac{dw}{w\sqrt{w}}\frac{T^2-\rho^2-C\rho w}{\sqrt{(T+\rho+Cw)(T-\rho-Cw)}}\,. \tag{4.13}$$

The integral diverges in the $w\to 0$ limit (near the boundary of AdS) and has been therefore cutoff at $u=1/\epsilon$. Performing the integration gives the following result for the energy:

$$\mathcal{E} = -L\sqrt{2TC}\Big[2\mathbf{E}(k)-\mathbf{K}(k)\Big]+\frac{2L}{\epsilon}\sqrt{T^2-\rho^2}\,. \tag{4.14}$$

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The divergent second term is independent of the distance between the string endpoints. It could be removed by adding a boundary term to the DBI action, and can be anyway considered as a renormalization of the 'mass' of the domain wall. Removing this divergent term, and plugging in the expression (4.10) for the integration constant, leads to the final expression for the renormalized energy:

$$\mathcal{E}_{\text{ren}} = -\frac{LT}{R}\Big[2\mathbf{E}(k) - \mathbf{K}(k)\Big]^2 . \tag{4.15}$$

Notice that it has the correct $1/R$ scaling behaviour of a Casimir energy. This is reassuring, though hardly surprising.

The really interesting story in the above expression is its non-trivial dependence on p and q. This is due to the fact that the brane has non-trivial coupling to the background flux. In the standard conventions in which the ratio of the F-string to the D-string tension is the string coupling, g_s, one finds for the argument of the elliptic integrals:

$$2k^2 = 1 - \frac{qg_s}{\sqrt{p^2 + g_s^2 q^2}} . \tag{4.16}$$

There are two instructive limits one can consider. First, the limit $q \to \infty$ (or equivalently $p \to 0$) where the brane is basically a collection of q pure fundamental strings, and $k \to 0$. In this limit, the Casimir energy reads

$$\mathcal{E}_{\text{ren}} = -\frac{\pi}{8LR}\, qQ_5 , \tag{4.17}$$

where we have used the relation between the background radius and the number of NS fivebranes, $L^2 = Q_5\alpha'$. This is the Casimir energy of a CFT with central charge $6qQ_5$, confined to an interval of size LR. We will explain in the following subsection why this agrees with the naive sigma-model expectation.

The second interesting limit, that of pure D-strings, is the natural starting point of a perturbative expansion at weak string coupling. From equation (4.16) we get:

$$k = \frac{1}{\sqrt{2}}\left[1 - \frac{qg_s}{2|p|} + o(g_s^3)\right] . \tag{4.18}$$

Expanding out the expression for the Casimir energy, and using the special values of the complete elliptic integrals at $k = \frac{1}{\sqrt{2}}$, one finds:

$$\mathcal{E}_{\text{ren}} = -\frac{2\pi^2}{\Gamma(\frac{1}{4})^4 LR}\frac{pQ_5}{g_s} - \frac{qQ_5}{4LR} + o(g_s) . \tag{4.19}$$

The leading term should be compared to the holographic Wilson loop computation in four-dimensional Yang-Mills theory. With our conventions of measuring the energy, the result for the quark/antiquark potential is [18, 19]

$$\mathcal{E}_{q\bar{q}} = -\frac{2\pi^2\sqrt{4\pi g_{\text{YM}}^2 N}}{\Gamma(\frac{1}{4})^4 LR} . \tag{4.20}$$

The two results are identical if one notes that the radius of AdS_5 is given (in string units) by $L^4 = 4\pi g_{\rm YM}^2 N$, whereas for AdS_3 it is determined by $L^2 = Q_5$. This is no surprise since both calculations minimize a pure tensive energy, which is proportional to the geometric length of the string. From the sigma-model point of view this Casimir energy is harder to understand, as we will explain in the next subsection. Note, finally, that the second term in the expansion (4.19) looks like a renormalized contribution to the central charge.

4.3 Symmetric product orbifolds and moduli flows

We will now consider this computation from the point of view of the space-time CFT. We will here make a series of remarks, leaving the more detailed comparison for future work. Before taking the near-horizon limit, the configuration is described by a string junction built out of the strings with charges $\vec{q}_1$, $\vec{q}_2$ and $\vec{q}_3 = \vec{q}_1 + \vec{q}_2$. We assume that the string $\vec{q}_1$, and therefore also the string $\vec{q}_3$, are much heavier than the string $\vec{q}_2$. Geometrically this implies that the "probe string" $\vec{q}_2$ is perpendicular to both $\vec{q}_1$ and $\vec{q}_3$, which are parallel.

We now take the usual AdS/CFT decoupling limit. From the bulk point of view we obtain the supergravity configuration of the previous section, where the light string $\vec{q}_2$ is treated as a probe brane. From the boundary point of view the two heavy strings $\vec{q}_1$ and $\vec{q}_3$ each flow to a conformal field theory in the infrared. The two conformal field theories are glued together along the string junction.

What is the fate of the string $\vec{q}_2$? Since we take the near-horizon limit in the direction perpendicular to the heavy strings, in this approximation there is no non-trivial decoupling limit of the light string $\vec{q}_2$. Its worldsheet excitations are in the perpendicular direction. Therefore in the IR limit holography dictates that the zero modes of the $\vec{q}_2$ string survive as moduli of the space-time CFT. There are no separate degrees of freedom living at the intersection point of the string junction. The junction is basically a junction of (p, q) strings in the background of fivebranes. A junction of (p, q) strings can be thought of as a single M-theory M2 brane wrapping a suitable one cycle of a two-torus. Since this is a smooth membrane configuration there should be no localized degrees of freedom at the intersection. Thus the final space-time theory consists of two CFT's on a half cylinder, separated by domain walls of the type we have been discussing so far.

It remains to discuss the way in which the two CFTs are glued together along the defect line. By general principle the CFT labeled by the charge $\vec{q}$ can be identified with a $\mathcal{N} = (4, 4)$ sigma model with target space M_N. Here M_N is a hyper-Kähler manifold that is a deformation of the symmetric product $S^N M = M^N / S_N$ with $N = \vec{q}^2/2$ for $M = T^4$ and $N = \vec{q}^2/2 + 1$ for $M = K3$. In general this deformation is determined by the charge vector $\vec{q}$ and the original moduli of the string theory background. It will include both metric deformations and sigma model B-fields. The metric deformations will include turning on twist fields in the orbifold description of the symmetric product. The CFT B-fields correspond to space-time RR backgrounds.

The naive supergravity dual will have $B = 0$ and will be strongly coupled, in the sense of both small target space volume and large twist field hyper-Kähler deformations. T-dualities do not in general suffice to relate small volume to large volume sigma models. The weakly coupled space-time CFT — the analogue of perturbative Yang-Mills theory in four dimension — is given by the orbifold CFT on $S^N M$ at large volume. In this regime the supergravity becomes a string theory with large RR fields (since $B_{CFT} = 1/2$) and large (ten-dimensional) string coupling constant.

Therefore, as always, the supergravity and weak-coupling CFT computations are in disjunct regimes. We will see that they indeed give qualitatively different behaviour for the domain walls. This is to be expected in view of the four-dimensional Wilson loop computations, where one observes a similar discrepancy. Alternatively, notice that a weakly-coupled CFT should have operators of arbitrary spin in its spectrum, and hence cannot be described by pure supergravity.

In the case of general charge vectors $\vec{q}_1$ and $\vec{q}_3 = \vec{q}_1 + \vec{q}_2$ the two CFTs will have different central charges and will be described by sigma models with target spaces of different topology M_{N_1} and M_{N_3}. In a semi-classical regime the gluing of the two sigma models will be given by a D-brane $Y \subset M_{N_1} \times M_{N_3}$. (Locally such a brane can be given by the graph of a function $\varphi:\ M_{N_1} \times M_{N_3}$. Globally, we are dealing with a generalized function, know mathematically as a correspondence.)

We will simplify now the discussion to the case where the emitted string is either a pure fundamental string or a pure D-string. In both cases we will compare the CFT with the supergravity computation.

4.3.1 Fundamental strings

Let us start with the case where the string $\vec{q}_2$ is a fundamental string. In this case we are always dealing with a bound state of Q_5 NS 5-branes and Q_1 fundamental strings. This system is dual to the famous D1-D5 system that has been studied extensively. In this case the space-time CFT is well-known. It is given by a sigma model on the target space $\mathcal{M}_{Q_5,Q_1}$ — the moduli space of charge Q_1 instantons in a $\mathrm{U}(Q_5)$ Yang-Mills theory on the four-manifold $M^4 = T^4, K3$. For relative prime (Q_1, Q_5) this moduli space is indeed a hyper-Kähler deformation of the symmetric product $S^N M$ with $N = Q_1 Q_5$.

We will be considering a string junction with $\vec{q}_1 = (Q_1, Q_5)$, $\vec{q}_2 = (q, 0)$ and $\vec{q}_3 = (Q_1 + q, Q_5)$. Physically the process whereby q fundamental strings are absorbed corresponds to addition of q extra pointlike instantons in the Yang-Mills theory. The gluing map

$$\varphi:\ \mathcal{M}_{Q_5,Q_1} \to \mathcal{M}_{Q_5,Q_1+q} \tag{4.21}$$

can be described informally as follows. Place q coincident pointlike instantons at a point x in the four-manifold M and add this solution to the smooth Q_1-instanton. This map depends on the choice of point $x \in M$. The map φ gives an isometric embedding of $\mathcal{M}_{Q_5,Q_1}$ into $\mathcal{M}_{Q_5,Q_1+q}$. This can be easily seen in a local computation of instantons on $\mathbb{R}^4$ where the ADHM construction can be used. We will give a more precise analysis of the geometry in section 4.3.3.

The corresponding D-brane that describes the gluing with the use of the folding construction is now given geometrically by the graph of the map φ in the product of the two instanton moduli spaces. It has dimension $4Q_1Q_5$.

Let us sketch now an argument why the Casimir energy of two such domain walls should be straightforward to compute in this regime. We have Neumann boundary conditions for

$4Q_1Q_5$ bosons and fermions. The remaining $4qQ_5$ bosons and fermions that describe the normal directions to $\mathcal{M}_{Q_5,Q_1}$ will have Dirichlet boundary conditions. Because of the isometric embedding there will be no jump in the CFT moduli, once we have canonically normalized the kinetic terms in the sigma models. So in this case the sole contribution to the Casimir energy will be the jump in the central charge

$$\Delta c = 6qQ_5 \,. \tag{4.22}$$

If we separate the two domain walls over a distance 2ℓ, this gives a Casimir energy

$$\mathcal{E} = -\frac{\pi}{48\ell}\Delta c = -\frac{\pi}{8\ell}qQ_5 \,. \tag{4.23}$$

This answer coincides with the supergravity computation (4.17) if we use $\ell = LR$ for the domain size.

4.3.2 D-strings

Let us now concentrate on the other limit where the absorbed string is a pure D-string with charge p. In this case the interpretation in terms of instanton moduli spaces is less clear. If we dualize the NS 5-brane to a D5-brane to obtain a gauge theory formulation, the addition of a D-string is equivalent to adding a fundamental string. This is represented by an electric flux tube in the gauge-theory instanton background. From the gauge-theory point of view this description of the CFT limit is not well understood.

In this case the string charge vectors $\vec{q}_1$ and $\vec{q}_3$ will satisfy

$$\vec{q}_1^{\,2} = \vec{q}_3^{\,2} \,. \tag{4.24}$$

Therefore the two sigma models have equal central charge and are in fact topologically isomorphic. Both are given by a deformation of the symmetric product $S^N M$. They only differ in the value of the deformation moduli. One way to understand this is that there is a U-duality transformation $U \in O(5, 5+n; \mathbb{Z})$ that maps $\vec{q}_1$ to $\vec{q}_3$

$$U(\vec{q}_1) = \vec{q}_3 \,. \tag{4.25}$$

By definition the transformation U does not leave the charge vector $\vec{q}_1$ invariant. Therefore it does not descend to a T-duality of the sigma model.

We can understand this change in the moduli as follows. We start with a string with charge $\vec{q}_1$. In the IR limit the moduli of the CFT are obtained from the moduli of the string theory background through the attractor formalism. That is, the scalars flow towards their fixed values at the horizon where they satisfy $\vec{q}_{1,R} = 0$.

Abstractly, if $\mathcal{N}$ represents the full string theory moduli space, then $\mathcal{N}$ contains a sublocus $\mathcal{N}_{\vec{q}_1}$ that represent the fixed scalars for the charge vector $\vec{q}_1$. The moment we absorb the D-string, the charge vector changes to $\vec{q}_3 = \vec{q}_1 + \vec{q}_2$, and no longer satisfies the fixed scalar condition. The moduli will now start to run along the attractor flow lines to the new fixed point locus $\mathcal{N}_{\vec{q}_3}$ where $\vec{q}_{3,R} = 0$.

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Note that the U-duality transformation U will map the fixed point locus $\mathcal{N}_{\vec{q}_1}$ to the fixed point locus $\mathcal{N}_{\vec{q}_3}$. We can therefore globally compare the values of the moduli of the two spacetime CFTs.

The flow in the moduli can be computed exactly using for instance the formalism developed by Mikhailov [42]. Here we just mention the first order effect in the D-brane charge p. The leading flow in the moduli is a contribution to the RR 0-form and 4-form fields.which are of the form

$$\delta(\text{RR-moduli}) \sim \frac{p}{g_s Q_1} \,. \tag{4.26}$$

At the symmetric product point we would expect that we can use a free field theory computation with order $Q_1 Q_5$ free fields. To get an idea what the answer will look like we can first do the calculation in case the target space is an n-dimensional torus with constant metric $G_{\mu\nu}$ and B-field $B_{\mu\nu}$. If we normalize B in such a way that it has integral periods, and assume that the domain wall separates two CFT's with equal metric, but with B-fields B and $B+\delta B$, the Casimir energy is proportional to

$$\mathcal{E} \sim -\frac{1}{L} G^{\alpha\beta} G^{\gamma\delta} \delta B_{\alpha\gamma} \delta B_{\beta\delta} \,. \tag{4.27}$$

If we apply the same equation to the symmetric product CFT, we obtain qualitatively the following result. Since the volume of $S^N M$ is of the form $f(Q_1,Q_5)V^N$, the Casimir energy must be of the form

$$\mathcal{E} \sim -\frac{1}{L} \frac{g(Q_1,Q_5)}{V} \left(\frac{p}{g_s Q_1}\right)^2 . \tag{4.28}$$

The fact that the B-field in question is dual to a two-cycle with self-intersection $Q_1 Q_5$ suggests that $g(Q_1,Q_5)$ is proportional to $Q_1 Q_5$ up to a factor of order unity, but a more careful analysis is required to make this precise. In any case, there will never be a precise agreement between the supergravity answer and the CFT calculation, because the first one is proportional to p, and the second one is proportional to p^2. This mismatch is similar to the disagreement found in $N=4$ SYM, where the supergravity answer is proportional to $\sqrt{g^2_{\mathrm{YM}} N}$, whereas the answer at weak coupling obtained in the gauge theory is proportional to $g^2_{\mathrm{YM}} N$.

4.3.3 Domain walls and Nakajima algebras

Let us first make a mathematical remark. If we have a map $f:\ X_1 \to X_2$ then there is of course the induced pull-back map $f^*:\ H^*(X_2) \to H^*(X_1)$ on the level of cohomology. Now the graph of the map f gives a subspace Y in the product space $X_1 \times X_2$ of points (x_1,x_2) that are related by $x_2 = f(x_1)$. In general if we just have a subspace Y in $X_1 \times X_2$ we speak of a correspondence instead of a map. Such a correspondence also gives rise to natural linear maps on the level of cohomology. More precisely, if π_1, π_2 are the projections of the product space on the two factors X_1, X_2, and if δ_Y denotes the cohomology class Poincaré dual to Y, then the maps are defined as

$$f_Y:\ H^*(X_2) \to H^*(X_1)\,, \qquad f_Y(\alpha) = \pi_{1,*}(\pi_2^* \alpha \wedge \delta_Y) \tag{4.29}$$

and the adjoint map is

$$f_Y^\dagger : \ H^*(X_1) \to H^*(X_2)\,, \qquad f_Y^\dagger(\beta) = \pi_{2,*}(\pi_1^*\beta \wedge \delta_Y)\,. \tag{4.30}$$

Here $\pi_{i,*}$ denotes the push-forward which is essentially integration over the fiber. The wedging with δ_Y restricts the differential form to the submanifold Y. In words what we do is lift the differential form to the product, restric to the subspace Y and subsequently project down to the other factor. One easily checks that the map f_Y reduces to f^* in the case that Y is the graph of a map f.

In the language of branes we can summarize this physically by saying that a D-brane Y in the supersymmetric sigma model on $X_1 \times X_2$ gives rise to a natural map between the Ramond ground states of the sigma models on X_1 and X_2. This can be interpretated in terms of domain walls if we represent them in the closed string channel. That is, if we think of the domain wall as an instantaneous brane on the worldsheet. In the closed string channel the domain wall becomes an operator, mapping incoming states in the sigam model on X_1 to outgoing states in the sigma model on X_2. We claim that at the level of ground states it represents exactly the induced map given by the correspondence Y.

These ideas can be applied in the case of where the AdS string is a pure fundamental string. In this case the string junctions made completey out of fivebranes and strings have an elegant interpretation in terms of the algebras studied in [20, 21, 22]. In fact we can even consider a more general situation where three strings join with charges $\vec{q}_1, \vec{q}_2, \vec{q}_3 = \vec{q}_1 + \vec{q}_2$, and where each string is built out of fivebranes and fundamental strings, i.e. the charge vectors are of the form $\vec{q}_i = (Q_5, Q_1)$. The folding construction that we have used previously to describe the junction of two CFTs can easily be extended to describe a junction of more than two CFTs. In that case the junction conditions are given in terms of a boundary state in the multiple tensor product of the corresponding Hilbert spaces. For sigma models that translates into a D-brane in the cartesian product of the target spaces.

For example, in the case of a three-string junction the domain wall will correpond to a boundary state in

$$|B\rangle\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3\,. \tag{4.31}$$

Each of the three world-sheet theories will flow in the IR to a sigma model with as target the instanton moduli space $\mathcal{M}_i$, $i = 1, 2, 3$. The junction is therefore geometrically, at large volume, given by a brane Y in the direct product

$$Y \subset \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3^*\,. \tag{4.32}$$

Here the asterix on $\mathcal{M}_3$ indicates that we choose minus the holomorphic symplectic form.

The "correspondence" Y has a mathematical interpretation when we represent the instanton parametrized by $\mathcal{M}_i$ in terms of a holomorphic vector bundle or more general a coherent torsion free sheaf $\mathcal{E}_i$. The locus Y is then given by triples $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ that are related by an exact sequence

$$0 \to \mathcal{E}_2 \to \mathcal{E}_3 \to \mathcal{E}_1 \to 0\,. \tag{4.33}$$

That is, the sheaf $\mathcal{E}_3$ can be obtained by an extension of $\mathcal{E}_1$ by $\mathcal{E}_2$ (or vice versa taking duals). The locus Y is a complex lagrangian submanifold in the given complex symplectic form. It therefore corresponds to a D-brane that preserves the diagonal $\mathcal{N}=4$ superconformal algebra in the tensor product CFT.

Note that more generally the D-brane Y is classified by an element of the K-theory group associated to the product $\mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3^*$. Domain walls in the 1-brane/5-brane system are therefore a natural place where the K-theory of instanton moduli spaces occurs within string theory.

As we explained, a correspondence of the form (4.32) naturally leads to linear maps on the level of the cohomology of the moduli spaces $\mathcal{M}_i$. More precisely in this case we get a map

$$\varphi: \ H^*(\mathcal{M}_1) \times H^*(\mathcal{M}_2) \rightarrow H^*(\mathcal{M}_3)\,. \tag{4.34}$$

This map given again by pull-back of a differential form on $\mathcal{M}_1 \times \mathcal{M}_2$ to the triple product, followed by restriction to the D-brane locus Y and push-forward (integrating over the fiber) to $\mathcal{M}_3$. In a formula

$$\varphi(\alpha,\beta) = \pi_{3,*}\left(\pi_1^*\alpha \wedge \pi_2^*\beta \cdot \delta_Y\right). \tag{4.35}$$

The adjoint is given by following this series of maps in the other direction.

If $\vec{q}_2 = (0,n)$ is built only out of strings and no fivebranes, the moduli space $\mathcal{M}_2$ parametrizes skyscraper sheaves that have there support at one point of the four-manifold M, so the moduli space is simply given by M itself. Therefore for every element $\alpha \in H^*(M)$ the map φ defined above reduces to a map

$$\alpha_n : H^*(\mathcal{M}_1) \rightarrow H^*(\mathcal{M}_3) \tag{4.36}$$

with

$$\mathcal{M}_1 = \mathcal{M}_{Q_5,Q_1}\,, \qquad \mathcal{M}_3 = \mathcal{M}_{Q_5,Q_1+n}\,. \tag{4.37}$$

Its adjoint $\alpha_{-n} = \alpha_n^\dagger$ is defined similarly. These maps have been studied extensively in the mathematical literature. In particular for the case $M = \mathbb{C}^2$ Nakajima [22] has shown that the operators α_n give rise to a Heisenberg algebra

$$[\alpha_n, \beta_m] = n\delta_{n+m,0} \int_M \alpha \wedge \beta\,. \tag{4.38}$$

We already remarked that these maps get a natural interpretation in the context of CFT domain walls that are the subject of this paper. Consider such a domain wall on the cylinder in the closed string channel labeled by some index I. That is, consider the domain wall along a space-like slice where it is interpreted as a euclidean instantaneous brane. Since this brane connects CFT_1 and CFT_3, it gives rise to a map on the level of Hilbert spaces

$$\varphi_I: \ \mathcal{H}_1 \rightarrow \mathcal{H}_3\,. \tag{4.39}$$

(This map will strictly speaking not exist at the level of proper Hilbert spaces since it will map normalizable states to unnormalizable states.) If we restrict the map φ_I to ground states we expect to find a generalization of Nakajima's map. This suggest that there is an

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interesting exchange algebra of such domain walls that should give rise to the commutation relations (4.38) in the quantum mechanics approximation. These relations have also been studied by Harvey and Moore in the context of the algebra of BPS states in [43]. It would be very interesting to connect these two points of view more directly.

To be concrete, we give the expression in the much simpler case of the free-field domain wall that we studied in section 2. For a given ϑ the corresponding operator

$$S_\vartheta : \mathcal{H} \to \mathcal{H}\,, \tag{4.40}$$

with $\mathcal{H}$ the free-boson Fock space, is given by (in the same canonical normalization as in section 2)

$$S_\vartheta = \prod_{n>0} \exp\left(-\frac{1}{n}\cos 2\vartheta\; a_n \bar{a}_n\right) \prod_{n>0} (\sin 2\vartheta)^{a_n a_{-n} + \bar{a}_n \bar{a}_{-n}} \prod_{n>0} \exp\left(-\frac{1}{n}\cos 2\vartheta\; a_{-n}\bar{a}_{-n}\right). \tag{4.41}$$

One easily verifies that for $\vartheta = \frac{\pi}{4}, \frac{\pi}{2}$ (that is $\lambda = 1, \infty$) this gives the correct expresion for a completely permeable, or completely reflective wall

$$S_{\frac{\pi}{4}} = \mathbf{1}\,, \qquad S_{\frac{\pi}{2}} = |0\rangle\langle 0|\,. \tag{4.42}$$

5. Outlook

An interesting problem for future work is to construct explicit models of permeable interfaces between strongly-coupled conformal field theories. As explained in section 3.2, one needs to find boundary states of tensor-product theories, which cannot be expressed in terms of Ishibashi states of the individual factors. One could try, for example, to embed WZW D-branes as 'non-factorizable' states in the $G/H \otimes H$ theory. Another place where to look for such defects is in the product of two WZW models with different Kac-Moody levels, for which K-theoretic arguments predict more charges than those that can be accounted for by elementary WZW D-branes.[8] Besides their intrinsic mathematical interest, such examples, if they exist, could find applications in condensed-matter physics.

Another natural question raised by our work is whether one can construct a string theory whose worldsheet contains permeable defects. One immediate difficulty with this idea is that if both CFT_1 and CFT_2 contain time coordinates in their target spaces, we need two Virasoro symmetries in order to remove all the negative-norm states from the spectrum. But the generic permeable walls only preserve one symmetry, as we have explained. One can try to circumvent this problem by asking, say, that CFT_2 have a euclidean target space. The no-ghost theorem requires, however, in this case that the total central charge of $\mathrm{CFT}_1 \otimes \mathrm{CFT}_2$ be 26. Thus, in the product theory one has a single time coordinate, a central charge 26, and a conventional conformal boundary state. This looks like a standard open-string theory on a regular D-brane! Whether there could be loopholes in the above argument is a question that deserves further thought.

[8] We thank Volker Schomerus for pointing out these arguments.

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A. Calculation of Casimir energy

In this appendix we calculate the Casimir energy for the set up described in section 2.2. We consider a real, massless scalar field $\tilde{\phi}$ in the interval $[-L, L]$, with Dirichlet boundary conditions at the endpoints. This is a choice of convenience that does not affect our final result. The action of $\tilde{\phi}$ is rescaled inside the subinterval $[-d, d]$, where $d < L$. This rescaling amounts to a change in radius, as discussed in the main text. The general plane-wave solution is of the form:

$$\phi(x,t) = e^{i\omega t} \times \begin{cases} A_1 \sin(\omega x + \delta_1) & \text{for } x \in [-L, -d] \\ A_2 \sin(\omega x + \delta_2) & \text{for } x \in [-d, d] \\ A_3 \sin(\omega x + \delta_3) & \text{for } x \in [d, L] \,. \end{cases} \tag{A.1}$$

The Dirichlet boundary conditions at $x = \pm L$ imply:

$$\delta_1 = \omega L \pmod{\pi} \qquad \text{and} \qquad \delta_3 = -\omega L \pmod{\pi} \,. \tag{A.2}$$

The gluing conditions (2.1) at the two domain walls, on the other hand, read:

$$\tan(-\omega d + \delta_1) = \lambda^2 \tan(-\omega d + \delta_2) \,, \tag{A.3}$$

and

$$\tan(\omega d + \delta_3) = \lambda^2 \tan(\omega d + \delta_2) \,. \tag{A.4}$$

Putting together (A.2), (A.3) and (A.4) leads to a transcendental equation for the allowed frequencies,

$$\tan\left[\omega(d - L)\right] = \lambda^2 \tan(\omega d + \delta_2) \qquad \text{with} \qquad \delta_2 = 0 \quad \left(\text{mod}\, \frac{\pi}{2}\right) . \tag{A.5}$$

We can solve this equation analytically in the limit $L \to \infty$ with d held fixed. Let us write

$$\omega_n \equiv \frac{\pi}{2L}(n - \epsilon_n) \,.$$

The 'unperturbed' spectrum, in the absence of walls, has $\epsilon_n = 0$. Assuming that ϵ_n stays bounded, so that we can neglect terms of $o(\epsilon_n/L)$ in the equation, we find:

$$\epsilon_n = \begin{cases} \left(\frac{2}{\pi}\right) \arctan\left[\lambda^2 \tan\left(\frac{n\pi d}{2L}\right)\right] - \frac{nd}{L} \pmod 2 & \text{for } n \text{ even}, \\ \left(\frac{2}{\pi}\right) \arctan\left[\lambda^{-2} \tan\left(\frac{n\pi d}{2L}\right)\right] - \frac{nd}{L} \pmod 2 & \text{for } n \text{ odd}. \end{cases} \tag{A.6}$$

We have chosen $\delta_2 = 0$ for even n, and $\delta_2 = \pi/2$ for odd n, so that ϵ_n vanishes when $\lambda = 1$ (no walls). We will also choose the branch of the arctangent such that $-1 \le \epsilon_n \le 1$. This ensures that ω_n is closest to its 'unperturbed' value (and is consistent with our assumption of bounded ϵ_n).

Since ω_n and ω_{-n} correspond to the same wavefunction, the Casimir energy reads:

$$\mathcal{E} = \sum_{n=1}^{\infty} \frac{1}{2}\omega_n \,. \tag{A.7}$$

The result is of course UV divergent, so we must perform the summation with great care. We will use the standard regularized formula (see for example [44]):

$$\sum_{n=1}^{\infty}(n-\alpha) = -\frac{1}{12} + \frac{1}{2}\alpha(1-\alpha)\,. \tag{A.8}$$

The trick is to pick $L = Nd/2$ for integer N (which we will send eventually to infinity). If the limit $L \to \infty$ exists it should not matter how we approach it. With this choice the frequency shifts are periodic:

$$\epsilon_n = \epsilon_{n+N}\,, \tag{A.9}$$

Expressing the arbitrary positive integer n as follows: $n = lN - k$ with $l = 1, \ldots, \infty$ and $k = 0, 1, \ldots, N-1$, we decompose the Casimir energy into N sums regularized separately as in (A.8).[9] The result after some algebraic rearrangements is

$$\mathcal{E} = -\frac{\pi}{48L} - \frac{\pi}{4d}\sum_{k=0}^{N-1}\frac{\epsilon_k}{N}\left(1 - \frac{2k}{N} - \frac{\epsilon_k}{N}\right). \tag{A.10}$$

The first term is the 'unperturbed' Casimir energy, which vanishes in the $L \sim N \to \infty$ limit. The term quadratic in ϵ_n is also subleading, so we may drop it in this limit, as well. For $0 \le k < N/2$, the shift ϵ_k is in the desired range (between -1 and 1) and we can perform the remaining sum in (A.10) as it stands. The other half-range, $N/2 \le k < N$, contributes an equal amount to the energy, as can be seen by changing variable to $\tilde{k} = N - k$, and using the fact that $\epsilon_{N-\tilde{k}} = -\epsilon_{\tilde{k}}$.

Defining finally the continuous variable $2y = \pi(1 - 2k/N)$, and using standard trigonometric identities, leads to the integral expression for the Casimir energy:

$$\mathcal{E} = \frac{1}{\pi^2 d}\int_0^{\pi/2} dy\, y\, \left[2y - \arctan(\lambda^2 \tan y) - \arctan(\lambda^{-2}\tan y)\right]. \tag{A.11}$$

[9] The reader can be reassured about this manipulation of divergent sums by checking, for instance, that the formal identity $\sum_1^{\infty} n = \sum_{k=0}^{N-1}\sum_{l=1}^{\infty} N(l - \frac{k}{N})$ stays valid after regularization of the n- and l-sums as in (A.8).

This formula passes several consistency checks: it vanishes for $\lambda = 1$, it has manifest symmetry under inversion of λ (which is equivalent to a T-duality transformation), and it gives the expected Casimir energy, $\mathcal{E} = -\pi/48d$, in the case of perfectly-reflecting walls ($\lambda = 0$).

We can perform the integral (A.11) explicitly by using the dilogarithm function $\mathrm{Li}_2(z)$. This has the series and integral representations (for $z < 1$)

$$\mathrm{Li}_2(z) = \sum_1^\infty \frac{z^m}{m^2} = -\int_0^z \frac{\log(1-w)}{w}\, dw\,. \tag{A.12}$$

Many of its properties can be found in ref. [23]. It obeys, in particular, the identity

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \frac{1}{2}\,\mathrm{Li}_2(z^2)\,. \tag{A.13}$$

We also need the integration formula [23]

$$\int_0^{\pi/2} \frac{y^2 dy}{1 - P\cos(2y)} = \frac{1+p^2}{1-p^2}\left[\frac{\pi^3}{24} + \frac{\pi}{2}\,\mathrm{Li}_2(-p)\right] \tag{A.14}$$

where

$$P = \frac{2p}{1+p^2}\,, \qquad \text{with } p^2 < 1\,. \tag{A.15}$$

Integrating the right-hand-side of (A.11) by parts, and using the above equations, puts the Casimir energy in the compact form quoted in the main text:

$$\mathcal{E} = -\frac{1}{8\pi d}\,\mathrm{Li}_2(\mathcal{R}^2) \qquad \text{with } \mathcal{R} = \frac{1-\lambda^2}{1+\lambda^2}\,. \tag{A.16}$$

Here $\mathcal{R}$ is the reflection coefficient. For total reflection $\mathcal{R} = \pm 1$, and since $\mathrm{Li}_2(1) = \pi^2/6$, we find indeed the standard Casimir energy of a scalar field. For weak reflection, the energy vanishes quadratically: $\mathcal{E} \simeq -\mathcal{R}^2/8\pi d$.

The dilogarithm function has appeared in CFT and integrable models, in various contexts (see for example [26]). The above interpretation as free-field Casimir energy is, to the best of our knowledge, new.

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